Einstein boundary conditions in relation to constraint propagation for the initial-boundary value problem of the Einstein equations

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We show how the use of the normal projection of the Einstein tensor as a set of boundary conditions relates to the propagation of the constraints, for two representations of the Einstein equations with vanishing shift vector: the ADM formulation, which is ill-posed, and the Einstein-Christoffel formulation, which is symmetric hyperbolic. Essentially, the components of the normal projection of the Einstein tensor that act as non-trivial boundary conditions are linear combinations of the evolution equations with the constraints that are not preserved at the boundary, in both cases. In the process, the relationship of the normal projection of the Einstein tensor to the recently introduced "constraint-preserving" boundary conditions becomes apparent.

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I. INTRODUCTION

The numerical integration of the initial-boundary value problem of the vacuuum Einstein equations is characterized, as is well known [1], by the presence of four differential equations that must be satisfied by the solution of the initial-value problem. These four additional equations are normally written in the form of constraints – that is: differential equations with no time derivatives of the fundamental variables–, and represent the vanishing of the projection of the Einstein tensor along the normal n^a to the time foliation, namely: $G_{ab}n^b = 0$. This atypical evolution problem has been a constant source of controversy from the onset of numerical implementations, because it is commonly found that the numerical solution of the initial-boundary value problem with constrained initial data fails to solve the constraints soon after the initial time. Earlier numerical simulations predominantly opted for a scheme that forced the constraints at every time slice (see, for instance, [2]). In this scheme, the numerical solution of the initial-boundary value problem at a given time step is used as a starting guess to generate a numerical solution of the constraint equations on the same time slice, which is subsequently used as data for the initial-boundary value problem for the next slice, and so on. In more recent years, this forced constrained scheme has been abandoned in favor of unconstrained simulations that disregard the constraint equations, except that the growth of the constraint violations is monitored and kept to a minimum by stopping the numerical simulations before the constraint violations grow too large (see, for instance, [3]).

The reason for the generically observed drifting of numerical simulations away from the constrained solution has yet to be pinned down and may turn out to be a combination of analytic and computational issues. Several analytical factors have been identified that may have a role to play in this respect, ranging from ill-posedness to asymptotic instability of the propagation of the constraints [4, 5]. In the face of the recurrence of uncontrolled numerical instabilities that prevent numerical simulations from maintaining accuracy for a time scale commensurate with the emission of strong gravitational waves from gravitational collapse, the importance of constraint violations in numerical simulations had widely been regarded as secondary until recently. In [9], the authors show, with sufficiently accurate numerical experiments, that the growth of constraint violations plays a key role in shortening the running time of numerical simulations even when other factors are kept within control. Understanding the growth of constraint violations and figuring out techniques to keep them in check have become issues of high priority in the goal to extend the run time of numerical simulations to a regime where gravitational radiation can be extracted.

What we show in this article is that the projection of the Einstein tensor along the normal e^a to the boundary of the simulation, $G_{ab}e^b$, is equivalent to the constraints that are not identically satisfied at the boundary by virtue of the initial-value problem alone. This is probably a generic fact, irrespective of the details of the formulation of the initial-boundary value problem of the 3+1 Einstein equations. We prove this fact in the case of the ADM formulation, which is ill posed but has well-posed constraint propagation, and also in the case of the Einstein-Christoffel formulation, which is well posed and has well-posed constraint propagation.

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This fact connects intimately two factors of seemingly independent effect to numerical simulations; i.e., the problem of constraint violations and the problem of proper boundary conditions. Intuitively, if boundary values travel inwardly at the speed of light (or any finite speed), then their effect is confined to the sector of the final time slice that lies in the causal future of the boundary surface, regardless of the extent of the bulk. Thus if the bulk is significantly larger than the run time in units of the speed of light, the boundary effects will be negligible *in the bulk.* As a result, it is commonly assumed that boundary effects can normally be kept in check by "pushing the boundaries out" and this has largely become the implicit rule in numerical simulations. (This is mathematically true only in the case of strongly hyperbolic formulations, though.) The constraint violations, on the other hand, appear to remain in most cases. The two problems would thus seem independent of each other (in fact, they are itemized separately in [9]).

On closer inspection, given any formulation of the initial-boundary value problem there are constraints that are identically satisfied in the simulated region by virtue of a proper choice of initial values, and there is a subset of constraints that are not satisfied in the entire simulated region by virtue of the initial values alone. In a minimallistic sense, it is these "nonpreserved" constraints that need to be controlled. Because, as we show here, these "nonpreserved" constraints are equivalent (up to evolution) to $G_{ab}e^b = 0$, and because $G_{ab}e^b = 0$, in turn, are proper and necessary boundary conditions for the initial-boundary value problem–as we have shown previously [6]–, it follows that by treating the boundaries properly one may get a handle on keeping the relevant constraint violations in check as a byproduct – a reasonable deal, by all means.

The article is organized as follows. The ADM case, which is the simplest and most transparent one in spite of its modest repertory of useful mathematical attributes, is dealt with first in Section II. As an example of strongly hyperbolic formulations, the Einstein-Christoffel case is developed in Sections III and IV. Both cases are treated with the simplifying assumption of vanishing shift vector in the terminology of the 3+1 split. The reader may assume that the complexity of the argument would increase sharply in the presence of a non-vanishing shift vector, although its general gist should run unchanged. This work develops issues that were anticipated in [6].

II. ADM FORMULATION WITH VANISHING SHIFT

Throughout the article we assume the following form for the metric of spacetime in coordinates $x^a = (x^i, t)$ in terms of the three-metric γ_{ij} of the slices at fixed value of t:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} dx^i dx^j \tag{1}$$

where α is the lapse function. The Einstein equations $G_{ab} = 0$ for the four-dimensional metric are equivalently expressed in the ADM form [1]:

$$\dot{\gamma}_{ij} = -2\alpha K_{ij}, \tag{2a}$$

$$\dot{K}_{ij} = \alpha\left(R_{ij} - 2K_{il}K^l{}_j + KK_{ij}\right) - D_i D_j \alpha, \tag{2b}$$

with the constraints

$$\mathcal{C} \equiv -\frac{1}{2}\left(R - K_{ij}K^{ij} + K^2\right) = 0, \tag{3a}$$

$$\mathcal{C}_i \equiv D_j K^j{}_i - D_i K = 0. \tag{3b}$$

Here an overdot denotes a partial derivative with respect to the time coordinate $(\partial/\partial t)$, indices are raised with the inverse metric γ^{ij}, D_i is the covariant three-derivative consistent with γ_{ij} , R_{ij} is the Ricci curvature tensor of γ_{ij}, R its Ricci scalar, K_{ij} is the extrinsic curvature of the slice at fixed value of t and $K \equiv \gamma^{ij} K_{ij}$. Expressed in terms of the Einstein tensor, the constraints (3) are related to specific components in the coordinates (x^i, t):

$$\mathcal{C} = -\alpha^2 G^{tt} \tag{4a}$$

$$\mathcal{C}_i = -\alpha \gamma_{ij} G^{jt}, \tag{4b}$$

where (4b) holds only for vanishing shift vector. The constraint character (the absence of second derivatives with respect to time) is a consequence of the fact that the only components of the Einstein tensor that appear in (4) have a contravariant index of value t. In geometric terms, (4) are linear combinations of $G_{ab}n^b = 0$ where n^b is the unit normal to the slices of fixed value of t, and is therefore given by $n^a = g^{ab} n_b = -\alpha g^{at} = \delta^a_t/\alpha$.

Similarly, at any boundary given by a fixed value of a spatial coordinate, the normal vector to the boundary surface e^b can be used to project the Einstein tensor as $G_{ab}e^b$ in order to obtain the components that have no second derivatives across the boundary. The vanishing of these can then be imposed as conditions on the boundary values for the fundamental variables. To fix ideas let's choose a boundary surface at a fixed value of x. Thus $e^b = g^{bx} = (0, \gamma^{ix})$ up to scaling. Consequently, $G_{ab}e^b = G^x_a$, so any linear combination of the components of the Einstein tensor with a contravariant index of value x will be suitable. Explicitly we have:

$$\begin{aligned} G^x_t = & -\frac{1}{2}\gamma^{ix}\left((\ln\gamma)_{,it} - \gamma^{kl}\dot{\gamma}_{ik,l}\right) - KD^x\alpha + K^x_k D^k\alpha \\ & +\alpha\left(\gamma^{kl}\Gamma^j{}_{kl}K^x_j + \gamma^{ix}\Gamma^j{}_{ik}K^k_j\right) \end{aligned} \tag{5a}$$

$$G^x_y = -\frac{\dot{K}^x_y}{\alpha} + KK^x_y + R^x_y - \frac{1}{\alpha}D^x D_y \alpha \tag{5b}$$

$$G^x_z = -\frac{\dot{K}^x_z}{\alpha} + KK^x_z + R^x_z - \frac{1}{\alpha}D^x D_z \alpha \tag{5c}$$

$$\begin{aligned} G^x_x = & \frac{\dot{K} - \dot{K}^x_x}{\alpha} - \frac{1}{2}(R + K^{ij}K_{ij} + K^2) + KK^x_x \\ & +R^x_x + \frac{1}{\alpha}\left(D^j D_j \alpha - D^x D_x \alpha\right) \end{aligned} \tag{5d}$$

Here $\Gamma^k{}_{ij} = (1/2)\gamma^{kl}(\gamma_{il,j} + \gamma_{jl,i} - \gamma_{ij,l})$, and the time derivative of the components of the extrinsic curvature is applied after raising an index, that is: $\dot{K}^i_j \equiv (\gamma^{ik} K_{kj})_{,t}$. The reader can verify that R^x_y and R^x_z do not involve second derivatives with respect to x of any of the variables and that the combination $R^x_x - \frac{1}{2}R$ doesn't either.

At this point the question arises as to whether $G_{ab}e^b$ vanish identically on the boundary for any solution of the evolution equations (2) with initial data satisfying (3). To start with, by inspection one can clearly see that G^x_y and G^x_z are exactly the evolution equations (3) for the mixed components K^x_y and K^x_z of the extrinsic curvature (namely: Eq. (2b) multiplied by γ^{ix} and evaluated at $j = y$ and $j = z$). Thus two of the four boundary equations are identically satisfied by the solution of the evolution equations (irrespective of the initial data). That is not the case with G^x_x. If one uses the evolution equations for K^x_x and for the trace K that follow from (2b) to eliminate the time derivatives from G^x_x, one is left with the Hamiltonian constraint $\mathcal{C}$. Finally, using the definition of the extrinsic curvature (2a) into the expression for G^x_t it is straightforward to see that $G^x_t = \alpha\gamma^{xi}\mathcal{C}_i$ (this is necessarily so in the case of vanishing shift because $G^x_t = g_{ta}G^{ax} = -\alpha^2 G^{tx}$ and inverting (4b) one has that $G^{tx} = -\alpha^{-1}\gamma^{xi}\mathcal{C}_i$). In summary, if we represent the evolution equations (2a) and (2b) in the form $\mathcal{E}^\gamma_{ij} = 0$ and $\mathcal{E}^K_{ij} = 0$ respectively by transferring all the terms from the right into the left-hand side, we have

$$G^x_t = \alpha\mathcal{C}^x + \frac{1}{2}\gamma^{xj}\gamma^{kl}(\partial_l\mathcal{E}^\gamma_{jk} - \gamma^{xl}\gamma^{jk}\partial_l\mathcal{E}^\gamma_{jk}) \quad (6a)$$

$$G^x_y = \frac{1}{\alpha}\gamma^{xj}\mathcal{E}^K_{jy} \quad (6b)$$

$$G^x_z = \frac{1}{\alpha}\gamma^{xj}\mathcal{E}^K_{jz} \quad (6c)$$

$$G^x_x = -\mathcal{C} + \frac{1}{\alpha}(\gamma^{kl}\mathcal{E}^K_{kl} - \gamma^{xj}\mathcal{E}^K_{xj}) \quad (6d)$$

which can essentially be expressed in the form

$$G^x_t \sim \alpha\mathcal{C}^x \quad (7a)$$

$$G^x_y \sim 0 \quad (7b)$$

$$G^x_z \sim 0 \quad (7c)$$

$$G^x_x \sim -\mathcal{C} \quad (7d)$$

where the symbol $\sim$ represents *equality up to terms proportional to the evolution equations.* Clearly the vanishing of G^x_t and G^x_x at the boundary is intimately connected with the propagation of $\mathcal{C}$ and $\mathcal{C}^x$ towards the boundary. On the other hand, $\mathcal{C}^y$ and $\mathcal{C}^z$ are not related to the boundary equations.

To have a sense for why this should be so, one may now look at the auxiliary system of propagation equations for the constraints as implied from (2). Taking a time derivative of the constraints and using the evolution equations we have

$$\dot{\mathcal{C}} = \alpha\partial^i\mathcal{C}_i + \ldots \quad (8a)$$

$$\dot{\mathcal{C}}_i = \alpha\partial_i\mathcal{C} + \ldots \quad (8b)$$

where $\ldots$ denote undifferentiated terms. This is a well-posed system of equations in the sense that it is *strongly hyperbolic* [7]. The characteristic speeds are 0 and $\pm\alpha$. With respect to the unit vector $\xi^i = \gamma^{ix}/\sqrt{\gamma^{xx}}$, normal to the boundary, the characteristic fields that travel with zero speed are $\mathcal{C}^y$ and $\mathcal{C}^z$. This is an indication that these constraints could be interpreted as "static", and that their vanishing is preserved by the evolution at all times after the initial slice, that is: they will vanish at the boundary by virtue of the evolution equations and the initial values (of course, only so long as the other constraints also vanish, due to the coupling in the undifferentiated terms in the constraint propagation equations). It makes sense, thus, that they do not relate at all with the boundary equations $G^x_a = 0$ and that two of the boundary equations are redundant (i.e., trivial).

The characteristic fields that travel with non-zero speeds $\pm\alpha$ are, respectively, $\mathcal{C}^\pm \equiv \mathcal{C} \pm \mathcal{C}^x/\sqrt{\gamma^{xx}}$. The characteristic field $\mathcal{C}+\mathcal{C}^x/\sqrt{\gamma^{xx}}$ is outgoing at the boundary, which means that its value is propagated from the initial slice to the boundary. If this constraint combination vanishes initially, then it should also vanish at the boundary. On the other hand, the characteristic field $\mathcal{C} - \mathcal{C}^x/\sqrt{\gamma^{xx}}$ is incoming at the boundary, which means that its value at the boundary is *unrelated* to the initial values prescribed in the interior. Thus, this combination of constraints is not vanishing at the boundary by virtue of the initial data and the evolution. This is a "non-preserved" constraint. This constraint must be enforced at the boundary by means of appropriate boundary conditions if the solution of the ADM equations (2) at the final time slice is to satisfy all four of the constraints. Since the two boundary equations $G^x_x = 0$ and $G^x_t = 0$ are directly related to the two characteristic constraints with non-vanishing speeds, imposing either one or any linear combination of them *except* $G^x_t + (\alpha/\sqrt{\gamma^{xx}})G^x_x = 0$ on the boundary is equivalent to imposing a linear relationship $\mathcal{C}^- = B\mathcal{C}^+$ (whith a constant B) as a boundary condition on the system of equations for the constraints. These are valid boundary conditions that preserve the well-posedness of the evolution of the constraints [7].

Thus, of the two equations $G^x_x = 0$ and $G^x_t = 0$, one of them is necessary to avoid constraint violations. The remaining one, which we can represent by the linear combination $G^x_t + (\alpha/\sqrt{\gamma^{xx}})G^x_x = 0$, is equivalent to a condition on the *outgoing constraint* $\mathcal{C}^+$. Therefore, it is redundant to the system of evolution of the constraints. At this point, however, it is not clear whether this remaining equation would be redundant as a boundary condition for the ADM evolution. The reason is that since the ADM equations are not strongly hyperbolic, the number of boundary conditions it requires is not known.

We are led to infer that some of the components of the projection of the Einstein tensor normally to the boundary are in direct correspondence with those constraints that are not automatically preserved by the evolution equations and the initial values. In this particular case of the ADM formulation, because the evolution equations

thenselves are not strongly hyperbolic, perhaps nothing else can be said about the role of the equations $G_{ab}e^b = 0$ as boundary conditions, except that failure to impose the nontrivial one leads to constraint violations with guaranteed certainty.

Yet the case of the ADM equations has been sufficiently direct and transparent to provide us with a guide for the much more complicated (but also much stronger) case of formulations of the Einstein equations that are strongly hyperbolic. In the next Section, a particular formulation is chosen for its simplicity and its close relationship to the ADM case.

III. BOUNDARY CONDITIONS FOR THE EC FORMULATION WITH VANISHING SHIFT

Most hyperbolic formulations of the Einstein equations require the introduction of additional variables in order to reduce the differential order in the spacelike coordinates from second to first. The additional variables are always a complete set of linearly independent combinations of the space-derivatives of γ_{ij}. This inevitably introduces an ambiguity: whereas indices labeling second-order derivatives are automatically symmetric (which affords plenty of convenient cancellations), the same terms written in terms of the new variables are not manifestly symmetric, the symmetry becoming a commodity that may or not be "turned on" by imposing yet additional constraints. For example, in second order one has, unambiguously, $\gamma_{ij,kl} - \gamma_{ij,lk} = 0$. However, if one defines $d_{ijk} \equiv \gamma_{ij,k}$, one is free to write the same expression either as 0 or as $d_{ijk,l} - d_{ijl,k}$. Undoubtedly, it makes a big difference to a numerical code to write it one way or the other, since they are *not* the same when it comes to the solution-generating process. This ambiguity is usually taken advantage of in order to write the Einstein equations in manifestly well-posed form. The cost is the loss of some transparency and the addition of constraints and, as we will see shortly, boundary conditions.

Consider the Einstein-Christoffel [EC] formulation of the 3+1 equations [8]. By defining first-order variables as the following 18 linearly independent combinations of the space-derivatives of the metric:

$$f_{kij} \equiv \Gamma_{(ij)k} + \gamma_{ki}\gamma^{lm}\Gamma_{[lj]m} + \gamma_{kj}\gamma^{lm}\Gamma_{[li]m}, \qquad (9)$$

and choosing the lapse function as $\alpha \equiv Q\sqrt{\gamma}$ with Q assumed arbitrarily prescribed a priori, the 3+1 equations can be put in the following equivalent form [9]:

$$\begin{aligned}
\dot{\gamma}_{ij} &= -2\alpha K_{ij} &(10a)\\
\dot{K}_{ij} + \alpha\gamma^{kl}\partial_l f_{kij} &= \alpha\{\gamma^{kl}(K_{kl}K_{ij} - 2K_{ki}K_{lj}) + \gamma^{kl}\gamma^{mn}(4f_{kmi}f_{[ln]j} + 4f_{km[n}f_{l]ij} - f_{ikm}f_{jln} \\
&\quad + 8f_{(ij)k}f_{[ln]m} + 4f_{km(i}f_{j)ln} - 8f_{kli}f_{mnj} + 20f_{kl(i}f_{j)mn} - 13f_{ikl}f_{jmn}) \\
&\quad - \partial_i\partial_j \ln Q - \partial_i \ln Q\partial_j \ln Q + 2\gamma_{ij}\gamma^{kl}\gamma^{mn}(f_{kmn}\partial_l \ln Q - f_{kml}\partial_n \ln Q) \\
&\quad + \gamma^{kl}[(2f_{(ij)k} - f_{kij})\partial_l \ln Q + 4f_{kl(i}\partial_{j)} \ln Q - 3(f_{ikl}\partial_j \ln Q + f_{jkl}\partial_i \ln Q)]\} &(10b)\\
\dot{f}_{kij} + \alpha\partial_k K_{ij} &= \alpha\{\gamma^{mn}[4K_{k(i}f_{j)mn} - 4f_{mn(i}K_{j)k} + K_{ij}(2f_{mnk} - 3f_{kmn})] \\
&\quad + 2\gamma^{mn}\gamma^{pq}[K_{mp}(\gamma_{k(i}f_{j)qn} - 2f_{qn(i}\gamma_{j)k}) + \gamma_{k(i}K_{j)m}(8f_{npq} - 6f_{pqn}) \\
&\quad + K_{mn}(4f_{pq(i}\gamma_{j)k} - 5\gamma_{k(i}f_{j)pq})] - K_{ij}\partial_k \ln Q \\
&\quad + 2\gamma^{mn}(K_{m(i}\gamma_{j)k}\partial_n \ln Q - K_{mn}\gamma_{k(i}\partial_{j)} \ln Q)\} &(10c)
\end{aligned}$$

with the constraints

$$\begin{aligned}
\mathcal{C} &\equiv -\frac{1}{2}\gamma^{ij}\gamma^{kl}\{2(\partial_k f_{ijl} - \partial_i f_{jkl}) + K_{ik}K_{jl} - K_{ij}K_{kl} \\
&\quad + \gamma^{mn}[f_{ikm}(5f_{jln} - 6f_{ljn}) + 13f_{ikl}f_{jmn} \\
&\quad + f_{ijk}(8f_{mln} - 20f_{lmn}]\} = 0 &(11a)\\
\mathcal{C}_i &\equiv -\gamma^{kl}\{\gamma^{mn}[K_{ik}(3f_{lmn} - 2f_{mnl}) - K_{km}f_{iln}] \\
&\quad + \partial_i K_{kl} - \partial_k K_{il}\} = 0 &(11b)\\
\mathcal{C}_{kij} &\equiv 2f_{kij} - 4\gamma^{lm}(f_{lm(i}\gamma_{j)k} - \gamma_{k(i}f_{j)lm}) - \partial_k\gamma_{ij} = 0 &(11c)
\end{aligned}$$

to be imposed only on the initial data. Here $\mathcal{C}_{ijk}$ represent the definition of the additional 18 first-order variables f_{kij} needed to reduce the ADM equations to full first-order form (they represent (9) inverted for $\gamma_{ij,k}$ in terms of f_{kij}). No other constraints are needed to pick a solution of the Einstein equations out of the larger set of solutions of the EC equations. A point that deserves to be made here is that (10b) is an exact transcription of the ADM evolution equation (2b), with no mixing of constraints into it. In other words: Eq. (10b) reduces exactly to Eq. (2b) if f_{kij} is substituted back in terms of γ_{ij} (assuming the same lapse condition).

We need to translate the boundary equations (5) in terms of the EC variables. Because of the ambiguity in writing second derivatives $\gamma_{ij,kl} = \gamma_{ij,lk} = (\gamma_{ij,kl} + \gamma_{ij,lk})/2$ in terms of first derivatives of f_{kij} there are, in principle, many different ways to write $G^x_a = 0$ in

terms of f_{kij}. Notwithstanding, among all these different possibilities, the admissible boundary equations are determined by the requirement that no $x-$derivatives of any variables may occur. In the following, we write the simplest expression of the boundary equations (5) up to undifferentiated terms.

We start with Eq. (5b). The terms with second derivatives arise from the combination $R^x_y - \gamma^{xk}\alpha_{,yk}/\alpha$ where $\alpha = Q\sqrt{\gamma}$ with arbitrary Q. Up to undifferentiated terms in the fundamental variables this combination is

$$R^x_y - \frac{\gamma^{xk}\alpha_{,yk}}{\alpha} = -\frac{1}{2}\gamma^{xm}\gamma^{kl}(\gamma_{ym,kl} - \gamma_{yl,km} - \gamma_{km,yl}) - \gamma^{xm}\gamma^{kl}\gamma_{kl,ym} + \dots \quad (12)$$

which can equivalently be represented as

$$R^x_y - \frac{\gamma^{xk}\alpha_{,yk}}{\alpha} = -\frac{1}{2}\gamma^{xm}\gamma^{kl}(\gamma_{ym,kl} - \gamma_{yl,km}) + \frac{1}{2}\gamma^{xm}\gamma^{kl}\partial_y\gamma_{km,l} - \gamma^{xm}\gamma^{kl}\partial_y\gamma_{kl,m} + \dots \quad (13)$$

where it is clear that in the last two terms $\gamma_{km,l}$ and $\gamma_{kl,m}$ can be substituted now directly in terms of f_{kij} via

$$\gamma_{ij,k} = 2f_{kij} - 4\gamma^{lm}(f_{lm(i}\gamma_{j)k} - \gamma_{k(i}f_{j)lm}) \quad (14)$$

yielding

$$\frac{1}{2}\gamma^{xm}\gamma^{kl}\partial_y\gamma_{km,l} = \partial_y(-3f^k{}_k{}^x + 4f^x{}_k{}^k) + \dots \quad (15)$$

$$\gamma^{xm}\gamma^{kl}\partial_y\gamma_{kl,m} = \partial_y(-4f^k{}_k{}^x + 6f^x{}_k{}^k) + \dots \quad (16)$$

Thus the last two terms clearly have no $x-$derivatives. By a relabeling of the dummy indices the remaining term is equivalent to

$$\gamma^{xm}\gamma^{kl}(\gamma_{ym,kl} - \gamma_{yl,km}) = (\gamma^{xm}\gamma^{kl} - \gamma^{xl}\gamma^{km})\gamma_{ym,kl} \quad (17)$$

One can see by inspection that the contribution of $k = x$ is identically vanishing, so we have, equivalently:

$$\begin{aligned}(\gamma^{xm}\gamma^{kl} - \gamma^{xl}\gamma^{km})\gamma_{ym,kl} &= (\gamma^{xm}\gamma^{yl} - \gamma^{xl}\gamma^{ym})\gamma_{ym,yl} + (\gamma^{xm}\gamma^{zl} - \gamma^{xl}\gamma^{zm})\gamma_{ym,zl} \\ &= \partial_y\left[(\gamma^{xm}\gamma^{yl} - \gamma^{xl}\gamma^{ym})\gamma_{ym,l}\right] + \partial_z\left[(\gamma^{xm}\gamma^{zl} - \gamma^{xl}\gamma^{zm})\gamma_{ym,l}\right] + \dots \quad (18)\end{aligned}$$

Substituting $\gamma_{ym,l}$ in terms of f_{kij} via (14) we have

$$\frac{1}{2}(\gamma^{xm}\gamma^{yl} - \gamma^{xl}\gamma^{ym})\gamma_{ym,l} = f^y{}_y{}^x - f^x{}_y{}^y - f^k{}_k{}^x + f^x{}_k{}^k + \dots \quad (19)$$

$$\frac{1}{2}(\gamma^{xm}\gamma^{zl} - \gamma^{xl}\gamma^{zm})\gamma_{ym,l} = f^z{}_y{}^x - f^x{}_y{}^z + \dots \quad (20)$$

Using all these in (13) we finally have

$$R^x_y - \frac{\gamma^{xk}\alpha_{,yk}}{\alpha} = \partial_z\left(f^x{}_y{}^z - f^z{}_y{}^x\right) + \partial_y\left(f^x{}_y{}^y - f^y{}_y{}^x + 3f^x{}_k{}^k - 2f^k{}_k{}^x\right) + \dots \quad (21)$$

The principal terms in $R^x_z - \gamma^{xk}\alpha_{,zk}/\alpha$ can, evidently, be handled in the same manner, so we have the following expressions for (5b) and (5c) explicit up to undifferentiated terms in the fundamental variables:

$$G^x_y = -\frac{\dot{K}^x_y}{\alpha} + \partial_z\left(f^x{}_y{}^z - f^z{}_y{}^x\right) + \partial_y\left(f^x{}_y{}^y - f^y{}_y{}^x - 3f^x{}_k{}^k + 2f^k{}_k{}^x\right) + \dots \quad (22)$$

$$G^x_z = -\frac{\dot{K}^x_z}{\alpha} + \partial_y\left(f^x{}_z{}^y - f^y{}_z{}^x\right) + \partial_z\left(f^x{}_z{}^z - f^z{}_z{}^x - 3f^x{}_k{}^k + 2f^k{}_k{}^x\right) + \dots \quad (23)$$

The calculation of the representation of G^x_x in terms of f_{kij} is much longer, but equally straightforward. In the following only the main steps are indicated. We start with the explicit expression of the Ricci components in terms of the metric:

$$R^x_x - \frac{1}{2}R = \frac{1}{2}(\gamma^{xm}\gamma^{kl} - \gamma^{xl}\gamma^{km})\gamma_{km,xl} - \frac{1}{2}(\gamma^{yl}\gamma^{km} - \gamma^{ym}\gamma^{kl})\gamma_{ym,kl} - \frac{1}{2}(\gamma^{zl}\gamma^{km} - \gamma^{zm}\gamma^{kl})\gamma_{zm,kl} + \dots \quad (24)$$

Expanding appropriately some of the contractions indicated (with the guidance that no second $x-$derivatives may remain) leads to a large number of cancellations, yielding the equivalent form:

$$R^x_x - \frac{1}{2}R = -\frac{1}{2}(\gamma^{zm}\gamma^{yl} - \gamma^{zl}\gamma^{ym})\gamma_{ym,zl} - \frac{1}{2}(\gamma^{ym}\gamma^{zl} - \gamma^{yl}\gamma^{zm})\gamma_{zm,yl} + \dots \quad (25)$$

where it is obvious that ∂_z and ∂_y can be pulled out of terms that are expressible in terms of f_{kij}. We have

$$\frac{1}{2}(\gamma^{zm}\gamma^{yl} - \gamma^{zl}\gamma^{ym})\gamma_{ym,l} = f^y{}_y{}^z - f^z{}_y{}^y + f^z{}_k{}^k - f^k{}_k{}^z + \dots \quad (26)$$

$$\frac{1}{2}(\gamma^{ym}\gamma^{zl} - \gamma^{yl}\gamma^{zm})\gamma_{zm,l} = f^z{}_z{}^y - f^y{}_z{}^z + f^y{}_k{}^k - f^k{}_k{}^y + \dots \quad (27)$$

Consequently we have

$$R^x_x - \frac{1}{2}R = \partial_z(f^z{}_y{}^y - f^y{}_y{}^z + f^k{}_k{}^z - f^z{}_k{}^k) + \partial_y(f^y{}_z{}^z - f^z{}_z{}^y + f^k{}_k{}^y - f^y{}_k{}^k) + \dots \quad (28)$$

The terms on second derivatives of the lapse in (5d) must be represented in terms of f_{kij} as well. They are:

$$\begin{aligned}\frac{1}{\alpha}(D^k D_k\alpha - D^x D_x\alpha) &= \gamma^{ym}\frac{\alpha_{,my}}{\alpha} + \gamma^{zm}\frac{\alpha_{,mz}}{\alpha} + \dots \\ &= \frac{1}{2}\gamma^{kl}\left(\gamma^{ym}\gamma_{kl,my} + \gamma^{zm}\gamma_{kl,mz}\right) \\ &\quad + \dots \\ &= \partial_y\left(3f^y{}_k{}^k - 2f^k{}_k{}^y\right) \\ &\quad + \partial_z\left(3f^z{}_k{}^k - 2f^k{}_k{}^z\right) \\ &\quad + \dots \end{aligned} \tag{29}$$

With (28) and (29), the boundary equation (5d) reads

$$\begin{aligned}G^x_x \;=\; &\frac{\dot K - \dot K^x_x}{\alpha} \\ &+\partial_z(f^z{}_z{}^z + 2f^z{}_x{}^x - f^{xz}{}_x + 3f^z{}_y{}^y - 2f^y{}_y{}^z) \\ &+\partial_y(f^y{}_y{}^y + 2f^y{}_x{}^x - f^{xy}{}_x + 3f^y{}_z{}^z - 2f^z{}_z{}^y) \\ &+ \dots \end{aligned} \tag{30}$$

Finally, it is quite straightforward to translate the boundary equation (5a) in terms of f_{kij}:

$$G^x_t = \dot f^x{}_k{}^k - \dot f^k{}_k{}^x + \dots \tag{31}$$

where $\dot f^k{}_i{}^j \equiv \partial_t(\gamma^{km} f_{mil}\gamma^{jl})$.

We thus have the four equations (31), (30), (22) and (23) to be considered as potential boundary conditions for the EC equations. Which ones are identically satisfied by virtue of the EC equations and the initial values, and which ones remain as boundary conditions is determined by the characteristic fields of the EC equations.

The EC equations are symmetric hyperbolic (and thus strongly hyperbolic as well [7]). With respect to the unit vector $\xi^i \equiv \gamma^{xi}/\sqrt{\gamma^{xx}}$ which is normal to the boundary $x = x_0$ for the region $x \leq x_0$ there are 18 "static" characteristic fields (the six γ_{ij}, the six $f^y{}_i{}^j$ and the six $f^z{}_i{}^j$) and 12 characteristic fields traveling at the speed of light, of which six are incoming:

$$^-U^j_i \equiv K^j_i - \frac{f^x{}_i{}^j}{\sqrt{\gamma^{xx}}} \tag{32}$$

and six are outgoing:

$$^+U^j_i \equiv K^j_i + \frac{f^x{}_i{}^j}{\sqrt{\gamma^{xx}}} \tag{33}$$

With regards to boundary values, those of the static and outgoing fields are completely determined by the initial values. Because the time derivatives of the incoming fields $^-U^y_z, {}^-U^x_x$ and the combination $^-U^y_y - {}^-U^z_z$ do not occur in any of the four boundary equations (31), (30), (22) or (23), the values of these three incoming fields on the boundary are completely *arbitrary*. However, the time derivatives of $^-U^x_y$ and $^-U^x_z$ occur in (22) and (23), respectively, so the boundary values of $^-U^x_y$ and $^-U^x_z$ are determined by (22) and (23) in terms of outgoing and static characteristic fields, as well as initial values. Finally, the time derivatives of the combination $^-U^y_y + {}^-U^z_z$ appear in (31) and in (30), as well as those of the corresponding ougoing counterpart $^+U^y_y + {}^+U^z_z$. This means that the two equations (31) and (30) are equivalent to one boundary prescription for an incoming field $(^-U^y_y + {}^-U^z_z)$ and one condition on an outgoing field $(^+U^y_y + {}^+U^z_z)$ which must be identically satisfied.

Thus, in the EC formulation of the Einstein equations, *three* of the four equations $G_{ab}e^b = 0$ are non trivial, as opposed to one in the ADM case.

IV. RELATION TO CONSTRAINT PROPAGATION IN THE EC FORMULATION WITH VANISHING SHIFT

The question that arises in the EC case is how the three non-trivial boundary conditions in the set $G_{ab}e^b = 0$ relate to the constraints. In analogy with the ADM case of Section II, we anticipate that in the EC case there will be (at most) three constraints that will be "non-preserved" at the boundary, and that they will be related to the three non-trivial components of $G_{ab}e^b = 0$ by linear combinations with the evolution equations. To prove this would be far from trivial, in principle, but can indeed be done explicitly by using the ADM case as a guide, as we show in the following.

We start by looking at the auxiliary system of propagation of the constraints. The reader should note, in the first place, that the addition of the 18 first-order constraints $\mathcal{C}_{kij}$ automatically raises the differential order of the system of propagation equations for the whole set of 22 constraints to second order: the evolution equations for the set $(\mathcal{C}, \mathcal{C}_i, \mathcal{C}_{kij})$ implied by the EC equations involve second space-derivatives of $\mathcal{C}_{kij}$. We have explicitly:

$$\dot{\mathcal{C}} = \alpha\partial^i\mathcal{C}_i + \dots \tag{34a}$$

$$\begin{aligned}\dot{\mathcal{C}}_i &= \frac{1}{2}\alpha\partial^k\left(\partial_i\mathcal{C}_{kl}{}^l - \partial_l\mathcal{C}_{ki}{}^l + \partial_k\mathcal{C}_{li}{}^l - \partial_k\mathcal{C}_{il}{}^l\right) \\ &\quad -\alpha\partial_i\mathcal{C} + \dots \end{aligned} \tag{34b}$$

$$\dot{\mathcal{C}}_{kij} = \dots \tag{34c}$$

where ... denote undifferentiated terms. The system of auxiliary evolution equations (34) for the 22 constraints is not manifestly well posed because of the presence of second-derivatives in the right-hand side (in fact, some times this indicates ill-posedness [10]). However, it can be reduced to first differential order in the usual manner, by adding new "constraint" variables that are space-derivatives of the constraints. In this case we can do with

$$\mathcal{C}_{lkij} \equiv \frac{1}{2}\left(\partial_l\mathcal{C}_{kij} - \partial_k\mathcal{C}_{lij}\right), \tag{35}$$

which enlarges the system (34) to 40 variables in all, and

casts it in the following form:

$$\begin{aligned}\dot{\mathcal{C}} &= \alpha\partial^i\mathcal{C}_i + \ldots && \text{(36a)}\\ \dot{\mathcal{C}}_i &= -\alpha\left(\partial_i\mathcal{C} + \partial^k\mathcal{C}_{lki}{}^l + \partial^k\mathcal{C}_{kil}{}^l\right) + \ldots && \text{(36b)}\\ \dot{\mathcal{C}}_{kij} &= \ldots && \text{(36c)}\\ \dot{\mathcal{C}}_{lkij} &= \alpha\left(\gamma_{ki}\partial_l\mathcal{C}_j + \gamma_{kj}\partial_l\mathcal{C}_i - \gamma_{li}\partial_k\mathcal{C}_j - \gamma_{lj}\partial_k\mathcal{C}_i\right) + \ldots && \text{(36d)}\end{aligned}$$

We digress slightly now from the main point to point out that, if one chooses to do so, the new "constraint" variables $\mathcal{C}_{lkij}$ can be expressed in terms of the fundamental variables of the EC system, in which case they read:

$$\begin{aligned}\mathcal{C}_{lkij} &= \partial_l(f_{kij} - 2\gamma^{nm}(f_{nm(i}\gamma_{j)k} - \gamma_{k(i}f_{j)nm}))\\ &\quad -\partial_k(f_{lij} - 2\gamma^{nm}(f_{nm(i}\gamma_{j)l} - \gamma_{l(i}f_{j)nm})), \quad (37)\end{aligned}$$

and turn out to be identical to the "integrability conditions" $0 = 1/2(\partial_l\gamma_{ij,k} - \partial_k\gamma_{ij,l})$. This is usually the way that such "constraint quantities" necessary to reduce the propagation of the constraints to first differential order are introduced by most authors, starting with Stewart in [11]. They are not to be interpreted as additional constraints to impose on the initial data, though. In fact, $\mathcal{C}_{lkij} = 0$ holds identically for any initial data satisfying $\mathcal{C}_{kij} = 0$. They are not to be used as substitutes for $\mathcal{C}_{kij} = 0$ either, because the vanishing of $\mathcal{C}_{kij}$ does not follow from the vanishing of $\mathcal{C}_{lkij}$. Indeed, if $\mathcal{C}_{lkij} = 0$ is satisfied by γ_{ij} and f_{kij}, then there exists a field h_{ij} such that $2f_{kij} - 4\gamma^{lm}(f_{lm(i}\gamma_{j)k} - \gamma_{k(i}f_{j)lm}) = \partial_k h_{ij}$, but it does not follow that $h_{ij} = \gamma_{ij}$. The introduction of the additional "constraints" $\mathcal{C}_{lkij} = 0$ is mostly a formal procedure to study the evolution of the constraints as functions of the point –not through the fundamental variables. In fact, one does not need to include all of $\mathcal{C}_{lkij}$ as additional variables in order to reduce the propagation of the constraints to first differential order, since one can see that only the contractions $\mathcal{C}_{lki}{}^l$ and $\mathcal{C}_{kil}{}^l$ appear in the equations, and these are 12 variables, so at least six of the 18 new variables $\mathcal{C}_{lkij}$ are entirely redundant.

The immediate benefit of introducing the first-order constraint variables is that the system (36) is well posed in the sense that it is strongly hyperbolic with characteristic speeds of 0 (multiplicity 34), $+\alpha$ (multiplicity 3) and $-\alpha$ (multiplicity 3). This means that, with respect to the (outer) boundary at $x = x_0$, three characteristic constraint variables are incoming, three are outgoing and all the others are "static". The six non-static characteristic constraints, which we denote by ${}^{\pm}\mathcal{Z}_i$ (with $+$ for outgoing and $-$ for incoming), are explicitly:

$$\begin{aligned}{}^{\pm}\mathcal{Z}_x &= \mathcal{C}_x \pm \frac{1}{\sqrt{\gamma^{xx}}}\left(\mathcal{C} + \mathcal{C}^{kx}{}_{xk} + \mathcal{C}^x{}_{xk}{}^k\right) && \text{(38a)}\\ {}^{\pm}\mathcal{Z}_y &= \mathcal{C}_y \pm \frac{1}{\sqrt{\gamma^{xx}}}\left(\mathcal{C}^{kx}{}_{yk} + \mathcal{C}^x{}_{yk}{}^k\right) && \text{(38b)}\\ {}^{\pm}\mathcal{Z}_z &= \mathcal{C}_z \pm \frac{1}{\sqrt{\gamma^{xx}}}\left(\mathcal{C}^{kx}{}_{zk} + \mathcal{C}^x{}_{zk}{}^k\right) && \text{(38c)}\end{aligned}$$

Because three of the characteristic constraints are incoming at the boundary, even if they are set to zero initially they will not be vanishing at the boundary by virtue of the evolution equations. In fact, the three incoming constraints must be prescribed at the boundary, either arbitrarily, or as functions of the outgoing constraints. That is: the problem of the propagation of the constraints, Eq. (36), requires three boundary conditions. Thus we have proven the first part of our claim: there are three constraints that are "non-preserved" at the boundary, in the same number as nontrivial boundary conditions for the EC equations, as anticipated.

The second part of the claim is to prove that the incoming constraints are related to the equations $G^x_a = 0$ by terms that are proportional to the evolution equations. The size of the problem as regards the number of variables makes this practically impossible unless one had a good guess as to what the linear combinations ought to be. Guided by the ADM case, we may expect that, through the evolution equations, G^x_x will be related to $\mathcal{C}$ and G^x_t will be related to $\mathcal{C}^x$, except for terms involving the new constraints $\mathcal{C}_{kij}$ or $\mathcal{C}_{lkij}$. We also expect that G^x_y and G^x_z will only be related to the new constraints, but not the hamiltonian nor vector constraints. Our ansatz is explicitly as follows:

$$\begin{aligned}G^x_t &\sim \alpha\mathcal{C}^x && \text{(39a)}\\ G^x_y &\sim \mathcal{C}^{kx}{}_{yk} + \mathcal{C}^x{}_{yk}{}^k && \text{(39b)}\\ G^x_z &\sim \mathcal{C}^{kx}{}_{zk} + \mathcal{C}^x{}_{zk}{}^k && \text{(39c)}\\ G^x_x &\sim \mathcal{C} + \mathcal{C}^{kx}{}_{xk} + \mathcal{C}^x{}_{xk}{}^k && \text{(39d)}\end{aligned}$$

which is equivalent to (6) except for terms whose occurrence is suggested by the combinations of constraints that can be expressed directly in terms of linear combinations of the characteristic constraints ${}^{\pm}\mathcal{Z}_i$.

The ansatz may be verified (or disproven) simply by comparing the right hand sides of equations (31), (22), (23) and (30) with (39) term by term under the assumption that every occurrence of a time derivative in (31), (22), (23) or (30) must be substituted in terms of space-derivatives by means of the evolution equations, that is:

$$\begin{aligned}\dot{K}^j_i &= -\alpha\partial_k f^k{}_i{}^j + \ldots && \text{(40a)}\\ \dot{f}^k{}_i{}^j &= -\alpha\partial_k K^j_i + \ldots && \text{(40b)}\end{aligned}$$

In particular, using (40a) to eliminate $\dot{K}^x_y$, for (22) we have

$$G^x_y \sim \partial_l f^x{}_y{}^l + \partial_y(2f^k{}_k{}^x - 3f^x{}_k{}^k) + \ldots \quad (41)$$

On the other hand, *by definition* from (37) we have

$$\mathcal{C}^{kx}{}_{yk} + \mathcal{C}^x{}_{yk}{}^k = \partial_l f^x{}_y{}^l + \partial_y(2f^k{}_k{}^x - 3f^x{}_k{}^k) + \ldots \quad (42)$$

Thus (39b) is *verified.* By the same argument substituting y with z, (39c) is similarly verified.

Next, using (40a) to eliminate $\dot{K} - \dot{K}^x_x (= \dot{K}^y_y + \dot{K}^z_z)$, for (30) we obtain:

$$\begin{aligned}G^x_x \sim\; & -\partial_x\left(f^x{}_y{}^y + f^x{}_z{}^z\right)\\ & +\partial_y\left(2f^y{}_x{}^x - f^x{}_x{}^y + 2f^y{}_z{}^z - 2f^z{}_z{}^y\right)\\ & +\partial_z\left(2f^z{}_x{}^x - f^x{}_x{}^z + 2f^z{}_y{}^y - 2f^y{}_y{}^z\right) + \ldots (43)\end{aligned}$$

On the other hand, directly by the definition (37) and the expression (11a) for the hamiltonian constraint we have

$$\begin{aligned}&\mathcal{C} + \mathcal{C}^{kx}{}_{xk} + \mathcal{C}^{x}{}_{xk}{}^{k} \\ &\quad= -\partial_x \left(f^x{}_y{}^y + f^x{}_z{}^z\right) \\ &\qquad +\partial_y \left(2f^y{}_x{}^x - f^x{}_x{}^y + 2f^y{}_z{}^z - 2f^z{}_z{}^y\right) \\ &\qquad +\partial_z \left(2f^z{}_x{}^x - f^x{}_x{}^z + 2f^z{}_y{}^y - 2f^y{}_y{}^z\right) + \ldots \quad (44)\end{aligned}$$

Thus also (39d) is verified.

Finally, using (40b) to eliminate the time derivatives of $f^k{}_i{}^j$ in terms of space derivatives of K_i^j, Eq. (31) reads

$$G_t^x \sim -\alpha(\partial^x K - \partial^k K_k^x) + \ldots \quad (45)$$

which is manifestly equal to $\alpha\mathcal{C}^x$, thus verifying (39a).

If we represent the evolution equations (10b) and (10c) in the form $\tilde{\mathcal{E}}_{ij} = 0$ and $\tilde{\mathcal{E}}_{kij} = 0$, respectively, by simply transferring all the terms in the right-hand side to the left, what we have shown is that the following relationships between the projection of the Einstein tensor, the constraints and the evolution equations hold in the EC formulation:

$$\begin{aligned}G_t^x &= \alpha\mathcal{C}^x + \tilde{\mathcal{E}}^x{}_k{}^k - \tilde{\mathcal{E}}^k{}_k{}^x && (46a)\\ G_y^x &= \mathcal{C}^{kx}{}_{yk} + \mathcal{C}^x{}_{yk}{}^k - \alpha^{-1}\tilde{\mathcal{E}}_y^x && (46b)\\ G_z^x &= \mathcal{C}^{kx}{}_{zk} + \mathcal{C}^x{}_{zk}{}^k - \alpha^{-1}\tilde{\mathcal{E}}_z^x && (46c)\\ G_x^x &= \mathcal{C} + \mathcal{C}^{kx}{}_{xk} + \mathcal{C}^x{}_{xk}{}^k + \alpha^{-1}\tilde{\mathcal{E}}_k^k - \alpha^{-1}\tilde{\mathcal{E}}_x^x && (46d)\end{aligned}$$

It may be objected that the proof is not complete because the undifferentiated terms of the equations have not been shown explicitly to be the same on both sides. We think that it is quite clear that a complete proof in that sense may not be feasible, but yet the consistency of the principal terms and the geometrical foundation of all terms on both sides, taken together, give a very strong indication that the equality will hold term by term.

The relationship to the characteristic constraints is found by "inverting" (38) in order to have the fundamental constraints in terms of the characteristic constraints:

$$\begin{aligned}\mathcal{C}_i &= \frac{1}{2}\left({}^+\mathcal{Z}_i + {}^-\mathcal{Z}_i\right) && (47a)\\ \mathcal{C}^{kx}{}_{yk} + \mathcal{C}^x{}_{yk}{}^k &= \frac{\sqrt{\gamma^{xx}}}{2}\left({}^+\mathcal{Z}_y - {}^-\mathcal{Z}_y\right) && (47b)\\ \mathcal{C}^{kx}{}_{zk} + \mathcal{C}^x{}_{zk}{}^k &= \frac{\sqrt{\gamma^{xx}}}{2}\left({}^+\mathcal{Z}_z - {}^-\mathcal{Z}_z\right) && (47c)\\ \mathcal{C} + \mathcal{C}^{kx}{}_{xk} + \mathcal{C}^x{}_{xk}{}^k &= \frac{\sqrt{\gamma^{xx}}}{2}\left({}^+\mathcal{Z}_x - {}^-\mathcal{Z}_x\right) && (47d)\end{aligned}$$

Thus, up to terms proportional to the evolution equations, imposing $G_a^x = 0$ is equivalent to imposing

$$\begin{aligned}\gamma^{xi}({}^+\mathcal{Z}_i + {}^-\mathcal{Z}_i) &= 0 && (48a)\\ {}^+\mathcal{Z}_y - {}^-\mathcal{Z}_y &= 0 && (48b)\\ {}^+\mathcal{Z}_z - {}^-\mathcal{Z}_z &= 0 && (48c)\\ {}^+\mathcal{Z}_x - {}^-\mathcal{Z}_x &= 0 && (48d)\end{aligned}$$

From our discussion in Section III it follows that (48b) and (48c) need to be imposed, and then either (48a) or (48d) or, in fact, any linear combination of them, as the third boundary condition for the EC equations. Remarkably, this is entirely consistent with the constraint propagation problem (36), since (48b), (48c) plus *one* linear combination of (48a) and (48d) turn out to be an admissible complete set of boundary conditions for the three incoming constraints in terms of the three outgoing constraints. By admissible boundary conditions we mean, for instance, maximally dissipative boundary conditions, which, as is known [7], for homogeneous problems like this one, essentially take the form of ${}^-U_i = L_i^j\,{}^+U_j$ with a rather general (bounded) matrix L_i^j.

In fact, the reader will have no difficulty to recognize in (48a), (48b) and (48c) the "constraint-preserving" boundary conditions of the Neumann type of [12] (with $\eta = 4$), where boundary conditions arising from constraint propagation are discussed in a linearized setting (and furthermore restricting η to the interval $0 < \eta < 2$.) This means that the set of three boundary equations

$$\begin{aligned}G_t^x &= 0,\\ G_y^x &= 0,\\ G_z^x &= 0,\end{aligned}$$

for the EC formulation –with G_t^x, G_y^x, G_z^x given by (5a), (5b) and (5c) written in terms of the variables f_{kij} in such a way that the principal terms occur as in (31), (22) and (23) respectively– constitute the *exact* (nonlinear) form of the so called constraint-preserving boundary conditions of the Neumann type that one would write for the case of the Einstein-Christoffel formulation by following the prescription in [12]. This result generalizes to three dimensions the prediction that we advanced in [13], where the constraint preserving boundary conditions were found to be identical to the projections of the Einstein tensor normal to the boundary for the EC equations with the restriction of spherical symmetry.

What needs to be made clear, however, is the question of why should the equations $G_a^x = 0$ imposed on the boundary enforce the constraints in the interior, given that they are only equivalent to the constraints at places where the evolution equations are satisfied. As a matter of fact, the evolution equations are *not* imposed on the boundary (that's why one needs boundary prescriptions). Therefore it is not true that the constraints are being enforced on the boundary. Still, they are enforced in the interior, which is the goal. The argument is the following. The vanishing of G_a^x on the boundary is, in a sense,"carried" by the incoming fields to the interior, where the evolution equations are actually satisfied. In the interior, then, one has both $G_a^x = 0$ and $\mathcal{E}_{ij} = \mathcal{E}_{ijk} = 0$. Therefore, by (46), the constraints related to G_a^x are enforced in the interior. Thus, by imposing $G_a^x = 0$ on the boundary, one enforces the constraints that would otherwise be violated outside of the domain of dependence of the initial surface.

V. STABILITY CONSIDERATIONS

The fact that three linearly independent combinations of $G^x_a = 0$ except $G^x_t + \gamma^{xi} G^x_i = 0$ yield necessary and sufficient boundary conditions on the incoming fundamental fields of the EC equations has importance in its own right: they ensure the uniqueness of solutions to the initial-boundary-value problem. Additionally, the fact that such boundary conditions are related to the non-static characteristic constraints by linear combinations with the evolution equations is entirely equivalent to a procedure of "trading" transverse space derivatives by time derivatives using the evolution equations, which is the procedure that has been used in [12] to obtain what the authors refer to as constraint-preserving boundary conditions. Which of the two meanings (linear combinations with evolution versus trading derivatives) to use is only a matter of taste. We (as the authors of [12] apparently do) interpret this fact as an indication that the boundary conditions of the evolution problem are equivalent to imposing vanishing values of the constraints on the boundary (up to linear combinations of the constraints among themselves).

The reader with an interest in numerical applications may be concerned with the question of whether any or all sets of Einstein boundary conditions (by which we mean three linearly independent combinations of $G^x_a = 0$ except $G^x_t + \gamma^{xi} G^x_i = 0$) preserve the well-posedness of the *initial* value problem represented by the EC equations. More precisely, with the question of which sets of Einstein boundary conditions ensure that the solution at the final time depends continuously on the initial data. This question has not been answered in the previous sections, as it lies beyond the scope of the present work, requiring a calculation by means of either energy methods (in the best case) or Laplace transform methods (in the most likely case), as described in [7].

However, preliminary results of relevance to this question already exist in the recent literature, and it is appropriate to mention them and their immediate consequences. In [14] the authors undertake a Laplace-transform type of study of the well-posedness of several boundary prescriptions including some choices of the Einstein boundary conditions for the standard EC system, as in the present article (as opposed to "generalized" EC as in [12]). In Section IV of [14] the authors conclude that, in the linearization around Minkowski spacetime, the choice of three boundary conditions as $G_{xt} = G_{xy} = G_{xz} = 0$ is in fact well posed for the case of standard EC as in the present article. Since the linearization of $G^x_t = G^x_y = G^x_z = 0$ around Minkowski space coincides with $G_{xt} = G_{xy} = G_{xz} = 0$, this in fact proves that, in the linearization around flat space, the three Einstein boundary conditions $G^x_t = G^x_y = G^x_z = 0$ constitute a well-posed set of boundary conditions for the (standard) EC formulation.

Moreover, by the present paper, these three Einstein boundary conditions represent the well-posed Neumann boundary conditions of the "constraint-preserving" scheme of [12] if one were to extend those authors' argument and lexicon to the case $\eta = 4$. In the constraint-preserving scheme of [12] the authors did not allow the parameter of the generalized EC system to take the value $\eta = 4$ required for the standard EC system, for the reason that the energy method of [12] works well only for symmetric hyperbolic formulations with symmetric hyperbolic constraint propagation, which is not the case for $\eta \geq 2$. As a result, in [12] the authors prove that $\eta < 2$ (necessary for symmetric hyperbolic constraint propagation) is sufficient for well-posed constraint-preserving boundary conditions, but they do not prove that $\eta < 2$ is necessary. On our part, by combining the results of [14] (in which the Laplace-transform method is used rather than energy methods) with our current article as indicated, we are indeed demonstrating that $\eta < 2$ is, in fact, *not* necessary in order to have well-posed constraint-preserving boundary conditions of the Neumann type (this fact is actually indicated implicitly – but unambiguously– in [14]). This also disproves, by counterexample, the erroneous belief that the propagation of the constraints needs to be symmetric hyperbolic (not just strongly hyperbolic) in order for such well-posed constraint preserving boundary conditions to exist. But more interestingly, we are providing the principal terms of the exact nonlinear boundary conditions, the non-principal ones being precisely indicated by the exact expression of the components of the Einstein tensor, namely by full substitution of the fundamental variables in Eqs. (5).

It also needs to be pointed out to the readers that in the same paper [14] the authors show that imposing the linear combination $G_{xx} - G_{tt} = 0$ in addition to $G_{xy} = G_{xz} = 0$ leads to an ill-posed problem in the linearization around Minkowski space. This is an indication that $G^x_t - \gamma^{xi} G^x_i = 0$ used with $G^x_y = G^x_z = 0$ may lead to an ill-posed initial-boundary-value problem in the nonlinear case as well, though an actual proof for the nonlinear case is still lacking. As terminology goes, readers should be made aware that the authors of [14] inappropriately associate the concept of Einstein boundary conditions exclusively with this set, without acknowledging that, in the way we have used it, the term refers to all linear combinations of $G^x_a = 0$, including the well-posed ones, and including those that, as we show, are equivalent to the "constraint-preserving" boundary conditions of [12] in the linearization.

VI. CONCLUDING REMARKS

What has been shown is that the vanishing of the projection of the Einstein tensor normal to the boundary acts as boundary conditions for the EC equations that automatically enforce the constraints that are otherwise not propagated, and that the set $G^x_t = G^x_y = G^x_z = 0$, in particular, leads to a well-posed initial-boundary-value

problem ideally suited for numerical evolution.

The reader can see that the concept of Einstein boundary conditions can be applied to *any* formulation of the initial-boundary value problem of the Einstein equations. The choice of the EC formulation in this work is merely done for convenience as the clearest illustration. We think it is reasonable to postulate that in the case of *any* strongly hyperbolic formulation of the Einstein equations the Einstein boundary conditions will include a set that will be equivalent to constraint-preserving boundary conditions of the Neumann type. But most numerical simulations today are currently done using formulations that are not strongly hyperbolic, in which case the question of well-posedness becomes irrelevant, yet the problem of identifying useful and consistent boundary conditions remains. For such formulations, the Einstein boundary conditions eliminate some guesswork on boundary values and may help control the constraint violations.

Given the explicit expressions (5), the Einstein boundary conditions $G_{ab}e^b = 0$ (up to linear combinations as indicated throughout) for any 3+1 formulation of the Einstein equations are found by simply expressing (5) in the chosen representation of fundamental variables. The only subtlety that must be taken care of is that the final form of $G_{ab}e^b = 0$ in the chosen variables must contain no derivatives across the boundary of the same order as the evolution equations. That such a form exists is guaranteed by the Bianchi identities. In all cases, the procedure to obtain the correct form of the Einstein boundary conditions in the chosen variables parallels what is done in the current work for the case of the EC formulation. In particular, with perhaps very little effort our method can almost certainly be used to show (or disprove) our claim that the constraint-preserving boundary conditions of the Neumann type for the generalized EC systems as in [12] are, indeed, Einstein boundary conditions as well.

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